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# PARTITION DIMENSION OF EXTENDED ZERO DIVISOR GRAPHS

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**Abstract:** The ordered partition $\Pi = \{S_1, S_2, ..., S_k\}$ of the vertices of the connected graph $G$ is a resolving partition, if for any vertex $x \in V$ with respect to the partition $\Pi$ is the vector $\zeta(x|\Pi) = (d(x, S_1), d(x, S_2), ..., d(x, S_k))$ where $d(x, S_j), 1 \leq j \leq k$ represents the distance between the vertex $x$ and the set $S_j$, is different for every pair of vertices and is denoted by $pd(G)$. The partition dimension is the minimum of $k$ for which there is a resolving partition. In this paper, we investigate the partition dimension of the extended zero divisor graphs of certain finite commutative rings.

**Keywords and Phrases:** Partition dimension, Extended zero divisor graph, Ring of integers, Commutative ring.

**2020 Mathematics Subject Classification:** 13M05, 05C12, 13A70.

## 1. Introduction

The concept of resolvability and metric dimension was first introduced by Slater in [16]. The development of this concept lead to many application to the arrangement of the fewest possible loran/sonar detecting devices in a network to enable the unique representation of each vertex's position in relation to the devices in the set. Given that the metric dimension can reflect the least number of landmarks that uniquely indicate a robot's position as it moves in a graph space, this invariant has a wide range of applications in the field of robotics.

The introduction of partition dimension by Chatrand et al. [3] came after the invention of metric dimension by Slater P. J. [16]. The development of partition dimension with standard results is given in [5]. These concepts can be used to solve pattern recognition and image processing issues, which sometimes require the usage of hierarchical data structures [11], as well as to the representation of chemical molecules in chemistry [4]. Similar concepts have been studied in [12], [13] and [14].

Throughout this paper, we denote $R$ as a finite commutative ring with unity. We denote $Z(R)^*$ and $U(R)$ to represent the set of non-zero divisors and set of unit elements of $R$ respectively. Let $Z(R)^*$ be the vertices of a graph and for any two distinct vertices $x$ and $y$ of $R$ there exist two positive integers $m$ and $n$ such that $x^m y^n = 0$ with $x^m \neq 0$ and $y^n \neq 0$ is known as the extended zero divisor graph denoted by $\Lambda(R)$. The ordered partition $\Pi = \{S_1, S_2, ..., S_k\}$ of the vertices of the connected graph $G$ is a resolving partition, if for any vertex $x \in V$ with respect to the partition $\Pi$ is the vector $\zeta(x|\Pi) = (d(x, S_1), d(x, S_2), ..., d(x, S_k))$ where $d(x, S_j), 1 \leq j \leq k$ represents the distance between the vertex $x$ and the set $S_j$, is different for every pair of vertices and is denoted by $pd(G)$. The partition dimension is the minimum of $k$ for which there is a resolving partition. Note that if $G$ is a nontrivial connected graph, then $pd(G) \leq \beta(G) + 1$. Moreover, if $G$ is nether a path or a complete graph with $n \geq 4$, then $3 \leq pd(G) \leq n - 1$. Refer [1], [6], [10], [19] for other basic definitions from ring theory and graph theory. In Section 2, we discuss the partition dimension of $\Lambda(R)$ for some rings isomorphic to $\mathbb{Z}_m$. In Section 3, we determine the partition dimension of $\Lambda(\mathbb{Z}_m)$ for $m = l_1^s l_2, l_1^s l_2^s$. In Section 4, we study the partition dimension of $\Lambda(\mathbb{Z}_m)$ for $m = l_1^s l_2 l_3, l_1^2 l_2^2 l_3$ where $l_1, l_2, l_3$ are primes and $s$ is any positive integer.

## 2. Partition Dimension of $\Lambda(R)$

In this Section, we discuss the finiteness of the partition dimension of the extended zero divisor graph when $R$ is a finite commutative ring. We also calculate the partition dimension for the ring of integers modulo $m$ for some particular cases.

**Observation 2.1.** By Lemma 2.2 of [5], if $\Pi$ is a resolving partition of $V(G)$, then no vertices from the same distance similar class belong to the same elements of $\Pi$.

**Theorem 2.1.** *Let $R$ be a ring. Then*

*(i) $pd(\Lambda(R))$ is finite if and only if $R$ is finite.*

*(ii)* $pd(\Lambda(R))$ *is undefined if and only if* $R$ *is an integral domain or* $R = \mathbb{Z}_4$ *or* $R = \frac{\mathbb{Z}_2[X]}{(X^2)}$.

**Proof.**

(i) The vertex set of $\Lambda(R)$ is finite if $R$ is a finite ring. On the other hand, let us assume that $pd(\Lambda(R))$ is finite. Consider the resolving partition for $\Lambda(R)$, $\Pi = \{S_1, S_2, ..., S_k\}$ with the cardinality $k, k \geq 0$. The distance between any two vertices of $\Lambda(R)$ belongs to the set $\{0, 1, 2, 3\}$ since the $diam(\Lambda(R)) \leq 3$.

The $k-$ vector $\zeta(v|\Pi) = (d(v, S_1), d(v, S_2), ..., d(v, S_k))$ represents $v$ with respect to $\Pi$, and each coordinate is a member of the set $\{0, 1, 2, 3\}$. Therefore, $\zeta(v|\Pi)$ has $4^m$ possible values. Furthermore, since $\zeta(v|\Pi)$ is distinct for every $v$ in $V(\Lambda(R))$. Consequently, $|V(\Lambda(R))| \leq 4^m$. Hence the result.

(ii) Since $2 \leq pd(\Lambda(R)) \leq n$ for a connected graph of order $n \geq 2$, $pd(\Lambda(R))$ is undefined if and only if $\Lambda(R)$ has an empty vertex set or a single vertex set.

**Theorem 2.2.** *Let* $R$ *be a ring. Then*

*(i)* $\Lambda(R)$ *is a path if and only if* $pd(\Lambda(R)) = 2$.

*(ii)* $\Lambda(R)$ *is a complete graph if and only if* $pd(\Lambda(R)) = n$

*(iii) If* $\Lambda(R)$ *is a cycle, then* $pd(\Lambda(R)) = 3$.

*(iv) If* $\Lambda(R)$ *is a star graph with* $n$ *vertices, then* $pd(\Lambda(R)) = n - 1$ *for* $n \geq 3$ .

*(v) If* $\Lambda(R)$ *is a complete bipartite graph with* $\Lambda(R) \cong K_{m,n}$, *then*

$$pd(\Lambda(R)) = \begin{cases} m+1, & \text{if } m = n \\ max\{m, n\}, & \text{if } m \neq n. \end{cases}.$$

*(vi) If* $\Lambda(R)$ *is a bipartite graph with* $\Lambda(R) \cong G_{m,n}$, *then*

$$pd(\Lambda(R)) \leq \begin{cases} m+1, & \text{if } m = n \\ max\{m, n\}, & \text{if } m \neq n. \end{cases}.$$

**Proof.** These results follows from Preposition 2.1,2.3 and Theorem 2.4 from [5].

**Proposition 2.3.** *Let* $l_1$ *and* $l_2$ *are prime numbers. Then*

*(i)* $pd(\Lambda(\mathbb{Z}_m)) = l_1 - 1$ *if* $m = 2l_1$ *for* $l_1 > 2$.

*(ii)* $pd(\Lambda(\mathbb{Z}_m)) = l_2 - 1$ *if* $m = l_1 l_2$ *for* $l_1 < l_2,\ l_1, l_2 > 2$.

*(iii)* $pd(\Lambda(\mathbb{Z}_m)) = 2(l_1 - 1)$ *if* $m = 4l_1$ *for* $l_1 > 2$.

*(iv)* $pd(\Lambda(\mathbb{Z}_m)) = |Z(\Lambda(\mathbb{Z}_m))^*|$ *if* $m = l_1^s$ *for* $l_1 \geq 2,\ s \geq 2$.

**Proof.**

(i) If $m = 2l_1$ and $l_1 > 2$, then $\Lambda(\mathbb{Z}_m)$ is a star graph with $l_1$ vertices. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1 - 1$.

(ii) If $m = l_1 l_2$ for $l_1 < l_2,\ l_1, l_2 > 2$, then $\Lambda(\mathbb{Z}_m)$ is a complete bipartite graph with partite sets of cardinality $l_1 - 1$ and $l_2 - 1$. Since $l_1 < l_2$, $pd(\Lambda(\mathbb{Z}_m)) = l_2 - 1$.

(iii) If $m = 4l_1$ for odd primes, it is clear that the vertices are partitioned into 2 distinct distance similar equivalence classes of $U_1 = \{l_1 s \mid 1 \leq s \leq 3\}, U_2 = \{2t \mid 2t < n \text{ and } t \neq l_1, t \in \mathbb{N}\}$ which forms a complete bipartite graph with partite sets of cardinality 3 and $2(l_1 - 1)$. Hence, by Theorem 2.2, $pd(\Lambda(\mathbb{Z}_m)) = 2(l_1 - 1)$.

(iv) If $m = l_1^s$ for $l_1 \geq 2,\ s \geq 2$, then $\Lambda(\mathbb{Z}_m)$ is a complete graph with $|Z(\Lambda(\mathbb{Z}_m))^*|$ vertices. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = |Z(\Lambda(\mathbb{Z}_m))^*|$.

## 3. Partition Dimension of $\Lambda(\mathbb{Z}_m)$ when $m$ is a product of 2 prime powers

Given that the overall graphs in those scenarios turn out to be a degree more complex, it is unclear how our methods may be extended to the product of more than three primes and so we limit our study to certain particular cases. Now, we compute the partition dimension of $\mathbb{Z}_m$ for $m = l_1^s l_2$ based on two conditions: $l_1 < l_2 < l_3$ and $l_1 > l_2$, as shown in the following results.

**Proposition 3.1.** *Let* $m = l_1^s l_2,\ l_1 > l_2$ *numbers with* $l_1 < l_2 < l_3, l_1, l_2 \geq 2, s \geq 2, m \neq 4l_1$, *then* $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1}(l_2 - 1)$.

**Proof.** If $m = l_1^s l_2,\ l_1 < l_2 < l_3$, then the vertices can be partitioned into 3 distance similar equivalence classes as

$U_1 = \{tl_1 < m \mid t \in \mathbb{N}\} \backslash \{tl_2 \mid t \in \mathbb{N}\}$,

$U_2 = \{tl_2 < m \mid t \in \mathbb{N}\} \backslash \{tl_1 \mid t \in \mathbb{N}\}$,

$U_3 = \{tl_1 l_2 < m \mid t \in \mathbb{N}\}$.

Let us denote $U_i = \{a_{[i][j]}\}, 1 \leq i \leq 3, 1 \leq j \leq |U_i|$. Consider the set of partition $\Pi = \{S_1, S_2, ..., S_{l_1^{s-1}(l_2-1)}\}$.

$S_1 = \{a_{[1][1]}, a_{[2][1]}\}$,

$S_2 = \{a_{[1][2]}, a_{[2][2]}\}$,

...

$S_{|U_2|} = \{a_{[1][|U_2|]}, a_{[2][|U_2|]}\}$,

$S_{|U_2|+1} = \{a_{[1][|U_2|+1]}, a_{[3][1]}\}$,
...
$S_{|U_2|+|U_3|} = \{a_{[1][|U_2|+|U_3|]}, a_{[3][|U_3|]}\}$,
...
$S_{l_1^{s-1}(l_2-1)} = \{a_{[1][l_1^{s-1}(l_2-1)]}\}$.

The partition representation of $a_{[i][j]}$ with respect to the set $\Pi$ is given by
$\zeta(a_{[1][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_2|}, \underbrace{1, 1, ..., 1}_{|U_3|}, 2, 2, ..., 2)$, $j^{th}$ coordinate is 0,
$\zeta(a_{[2][j]}|\Pi) = (1, 1, ..., 1)$, $j^{th}$ coordinate is 0,
$\zeta(a_{[3][j]}|\Pi) = (1, 1, ..., 1)$, $(j + |U_2|)^{th}$ coordinate is 0.

By the above representation, $\zeta(a_{[i][j]}|\Pi)$ has a unique partition representation. Hence, $\Pi$ is a resolving partition and $pd(\Lambda(\mathbb{Z}_m)) \leq l_1^{s-1}(l_2 - 1)$.

Since $|U_1| = l_1^{s-1}(l_2 - 1)$ and by Observation 2.1, $pd(\Lambda(\mathbb{Z}_m)) \geq l_1^{s-1}(l_2 - 1)$. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1}(l_2 - 1)$ and this proves the result.

**Proposition 3.2.** *Let $m = l_1^s l_2$, $l_1 > l_2$ numbers with $l_1 > l_2, l_1, l_2 \geq 2, k \geq 2, m \neq 4l_1$, then $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1}(l_1 - 1)$.*

**Proof.** If $m = l_1^s l_2, l_1 > l_2$, then the vertices can be partitioned into 3 distance similar equivalence classes as
$U_1 = \{tl_2 < m \mid t \in \mathbb{N}\} \backslash \{tl_1 \mid t \in \mathbb{N}\}$,
$U_2 = \{tl_1 < m \mid t \in \mathbb{N}\} \backslash \{tl_2 \mid t \in \mathbb{N}\}$,
$U_3 = \{tl_1 l_2 < m \mid t \in \mathbb{N}\}$.
Let us denote $U_i = \{a_{[i][j]}\}, 1 \leq i \leq 3, 1 \leq j \leq |U_i|$. Consider the set of partition $\Pi = \{S_1, S_2, ..., S_{l_1^{s-1}(l_1-1)}\}$.
$S_1 = \{a_{[1][1]}, a_{[2][1]}\}$,
$S_2 = \{a_{[1][2]}, a_{[2][2]}\}$,
...
$S_{|U_2|} = \{a_{[1][|U_2|]}, a_{[2][|U_2|]}\}$,
$S_{|U_2|+1} = \{a_{[1][|U_2|+1]}, a_{[3][1]}\}$,
...
$S_{|U_2|+|U_3|} = \{a_{[1][|U_2|+|U_3|]}, a_{[3][|U_3|]}\}$,
...
$S_{l_1^{s-1}(l_1-1)} = \{a_{[1][l_1^{s-1}(l_1-1)]}\}$.

The partition representation of $a_{[i][j]}$ with respect to the set $\Pi$ is given by
$\zeta(a_{[1][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_2|}, 2, 2, ..., 2)$, $j^{th}$ coordinate is 0,
$\zeta(a_{[2][j]}|\Pi) = (1, 1, ..., 1)$, $j^{th}$ coordinate is 0,

$\zeta(a_{[3][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_2|},\underbrace{1,1,...,1}_{|U_3|},2,2,...,2),\ (j+|U_2|)^{th}$ coordinate is 0.

By the above representation, $\zeta(a_{[i][j]}|\Pi)$ has a unique partition representation. Hence, $\Pi$ is a resolving partition and $pd(\Lambda(\mathbb{Z}_m)) \leq l_1^{s-1}(l_1-1)$.

Since $|U_1| = l_1^{s-1}(l_1-1)$ and by Observation 2.1, $pd(\Lambda(\mathbb{Z}_m)) \geq l_1^{s-1}(l_1-1)$. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1}(l_1-1)$ and this proves the result.

The upcoming theorem shows the partition dimension of $\mathbb{Z}_m$ for $m = l_1^s l_2^s$.

**Proposition 3.3.** *Let $m = l_1^s l_2^s$, $l_1 > l_2$ with $l_1 < l_2, l_1, l_2 \geq 2$, then $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1} l_2^{s-1}(l_2-1)$.*

**Proof.** For $m = l_1^s l_2^s$, the vertices can be partitioned into $3s-1$ distance similar equivalence classes as

$U_1 = \{tl_1 < m \mid t \in \mathbb{N}\}\backslash\{tl_2 \mid t \in \mathbb{N}\}$,
$U_2 = \{tl_2 < m \mid t \in \mathbb{N}\}\backslash\{tl_1 \mid t \in \mathbb{N}\}$,
$U_3 = \{tl_1 l_2 < m \mid t \in \mathbb{N}\}\backslash\bigcup_{i=4}^{3s-1} U_i$,
$U_4 = \{tl_1^2 l_2 < m \mid t \in \mathbb{N}\}\backslash\bigcup_{i=5}^{3s-1} U_i$,
$U_5 = \{tl_1 l_2^2 < m \mid t \in \mathbb{N}\}\backslash\bigcup_{i=6}^{3s-1} U_i$,
$U_6 = \{tl_1^2 l_2^2 < m \mid t \in \mathbb{N}\}\backslash\bigcup_{i=7}^{3s-1} U_i$,
$U_7 = \{tl_1^3 l_2^2 < m \mid t \in \mathbb{N}\}\backslash\bigcup_{i=8}^{3s-1} U_i$,
$U_8 = \{tl_1^2 l_2^3 < m \mid t \in \mathbb{N}\}\backslash\bigcup_{i=9}^{3s-1} U_i$,
$U_9 = \{tl_1^3 l_2^3 < m \mid t \in \mathbb{N}\}\backslash\bigcup_{i=10}^{3s-1} U_i$,
...,
$U_{3s-3} = \{tl_1^{s-1} l_2^{s-1} < m \mid t \in \mathbb{N}\}\backslash\bigcup_{i=3s-2}^{3s-1} U_i$,
$U_{3s-2} = \{tl_1^{s} l_2^{s-1} < m \mid t \in \mathbb{N}\}\backslash U_{3s-1}$,
$U_{3s-1} = \{tl_1^{s-1} l_2^{s} < m \mid t \in \mathbb{N}\}$.
Let us denote $U_i = \{a_{[i][j]}\}, 1 \leq i \leq 3s-1, 1 \leq j \leq |U_i|$.

From Figure 1, we can see that the distance similar class $U_1$ is connected to the distance similar classes $U_5, U_8, U_{11}, ..., U_{3s-1}$ and the distance similar class $U_2$ is connected to the distance similar classes $U_4, U_7, U_{10}, ..., U_{3s-2}$ and the distance similar classes from $U_3$ to $U_{3s-1}$ are connected to one another and themselves.
Now, Consider the set of partition $\Pi = \{S_1, S_2, ..., S_{l_1^{s-1} l_2^{s-1}(l_2-1)}\}$.
$S_1 = \{a_{[1][1]}, a_{[2][1]}\}$,
$S_2 = \{a_{[1][2]}, a_{[2][2]}\}$,
...
$S_{|U_2|} = \{a_{[1][|U_2|]}, a_{[2][|U_2|]}\}$,
$S_{|U_2|+1} = \{a_{[1][|U_2|+1]}, a_{[3][1]}\}$,
...
$S_{|U_2|+|U_3|} = \{a_{[1][|U_2|+|U_3|]}, a_{[3][|U_3|]}\}$,

$S_{|U_2|+|U_3|+1} = \{a_{[1][|U_2|+|U_3|+1]}, a_{[4][1]}\},$
...
$S_{l_1^{s-1}l_2^{s-1}(l_2-1)} = \{a_{[1][l_1^{s-1}l_2^{s-1}(l_2-1)]}\}.$

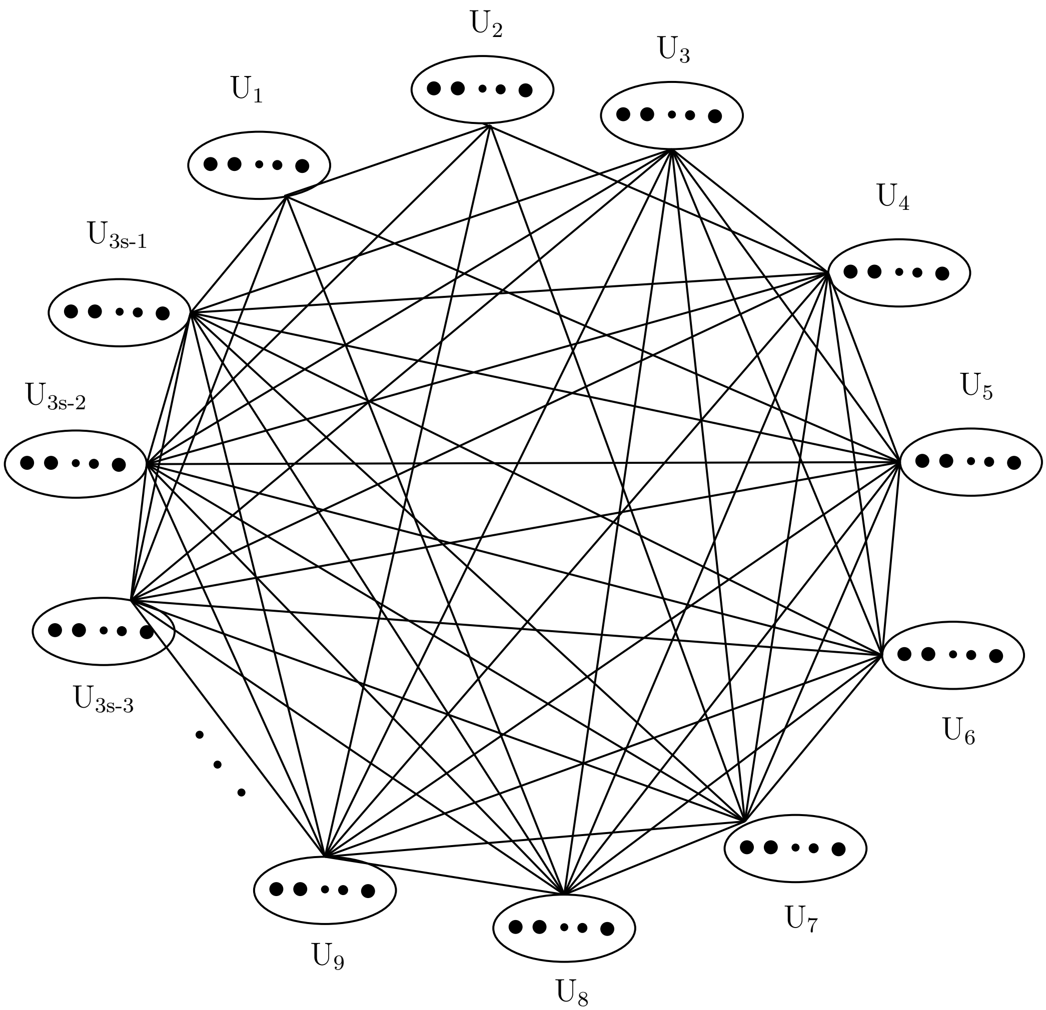


Figure 1: $\mathbb{E}(\mathbb{Z}_{l_1^s l_2^s})$

The partition representation of $a_{[i][j]}$ with respect to the set $\Pi$ is given by

$$\zeta(a_{[1][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_2|}, \underbrace{2,2,...,2}_{|U_3|+|U_4|}, \underbrace{1,1,...,1}_{|U_5|}, \underbrace{2,2,...,2}_{|U_6|+|U_7|}, \underbrace{1,1,...,1}_{|U_8|}, ..., \underbrace{2,2,...,2}_{|U_{3s-3}|+|U_{3s-2}|}, \underbrace{1,1,...,1}_{|U_{3s-1}|}, 2,2,...,2), \ j^{th} \text{ coordinate is } 0,$$

$\zeta(a_{[2][j]}|\Pi) = (1,1,...,1),\ j^{th}$ coordinate is 0,

For $i = 3k-1, k \geq 2$,
$\zeta(a_{[3k-1][j]}|\Pi) = (1, 1, ..., 1)$, $(j + \sum_{i=2}^{3k-2} |U_i|)^{th}$ coordinate is 0,
For $i = 3k, k \geq 1$,
$\zeta(a_{[3k][j]}|\Pi) = (\underbrace{2, 2, ..., 2}_{|U_2|}, \underbrace{1, 1, ..., 1}_{\sum_{i=3}^{3s-1}|U_i|}, 2, 2, ..., 2)$, $(j + \sum_{i=2}^{3k-2} |U_i|)^{th}$ coordinate is 0,
For $i = 3k+1, k \geq 1$,
$\zeta(a_{[3k+1][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{\sum_{i=2}^{3s-1}|U_i|}, 2, 2, ..., 2)$, $(j + \sum_{i=2}^{3k} |U_i|)^{th}$ coordinate is 0.

By the above representation, $\zeta(a_{[i][j]}|\Pi)$ has a unique partition representation. Hence, $\Pi$ is a resolving partition and $pd(\Lambda(\mathbb{Z}_m)) \leq l_1^{s-1} l_2^{s-1}(l_2 - 1)$.

Since $|U_1| = l_1^{s-1} l_2^{s-1}(l_2 - 1)$ and by Observation 2.1, $pd(\Lambda(\mathbb{Z}_m)) \geq l_1^{s-1} l_2^{s-1}(l_2 - 1)$. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1} l_2^{s-1}(l_2 - 1)$ and this proves the result.

## 4. Partition Dimension of $\Lambda(\mathbb{Z}_m)$ when $m$ is a product of 3 prime powers

In the following results, we calculate the partition dimension of $\mathbb{Z}_m$ for $m = l_1^s l_2 l_3$ using two conditions when $l_1 < l_2 < l_3$ and $l_1 > l_2 > l_3$.

**Proposition 4.1.** *Let $m = l_1^s l_2 l_3$ where $l_1, l_2$ and $l_3$ are prime numbers with $l_1 < l_2 < l_3, l_1, l_2, l_3 \geq 2, s \geq 2$, then $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1}(l_2 - 1)(l_3 - 1)$.*

**Proof.** If $m = l_1^s l_2 l_3, l_1 < l_2 < l_3$, then the vertices can be partitioned into 7 distance similar equivalence classes as
$U_1 = \{tl_1 < m \mid t \in \mathbb{N}\} \setminus \bigcup_{i=2}^{7} U_i$,
$U_2 = \{tl_2 < m \mid t \in \mathbb{N}\} \setminus \bigcup_{i=1, i\neq 2}^{7} U_i$,
$U_3 = \{tl_3 < m \mid t \in \mathbb{N}\} \setminus \bigcup_{i=1, i\neq 3}^{7} U_i$,
$U_4 = \{tl_1 l_2 < m \mid t \in \mathbb{N}\} \setminus U_7$,
$U_5 = \{tl_1 l_3 < m \mid t \in \mathbb{N}\} \setminus U_7$,
$U_6 = \{tl_2 l_3 < m \mid t \in \mathbb{N}\} \setminus U_7$,
$U_7 = \{tl_1 l_2 l_3 < m \mid t \in \mathbb{N}\}$.
Let us denote $U_i = \{a_{[i][j]}\}, 1 \leq i \leq 7, 1 \leq j \leq |U_i|$. Consider the set of partition $\Pi = \{S_1, S_2, ..., S_{l_1^{s-1}(l_2-1)(l_3-1)}\}$.
$S_1 = \{a_{[1][1]}, a_{[2][1]}, a_{[3][1]}, a_{[4][1]}\}$,

$S_2 = \{a_{[1][2]}, a_{[2][2]}, a_{[3][2]}, a_{[4][2]}\}$,
...
$S_{|U_4|} = \{a_{[1][|U_4|]}, a_{[2][|U_4|]}, a_{[3][|U_4|]}, a_{[4][|U_4|]}\}$,
$S_{|U_4|+1} = \{a_{[1][|U_4|+1]}, a_{[2][|U_4|+1]}, a_{[3][|U_4|+1]}, a_{[5][1]}\}$,
...
$S_{|U_4|+|U_5|} = \{a_{[1][|U_4|+|U_5|]}, a_{[2][|U_4|+|U_5|]}, a_{[3][|U_4|+|U_5|]}, a_{[5][|U_5|]}\}$,
...
$S_{l_1^{s-1}(l_2-1)(l_3-1)} = \{a_{[1][l_1^{s-1}(l_2-1)(l_3-1)]}\}$.

The partition representation of $a_{[i][j]}$ with respect to the set $\Pi$ is given by
$\zeta(a_{[1][j]}|\Pi) = (\underbrace{2,2,...,2}_{|U_4|}, \underbrace{2,2,...,2}_{|U_5|}, \underbrace{1,1,...,1}_{|U_6|}, \underbrace{1,1,...,1}_{|U_7|}, 2,2,...,2)$,
$j^{th}$ coordinate is 0,
$\zeta(a_{[2][j]}|\Pi) = (\underbrace{2,2,...,2}_{|U_4|}, \underbrace{1,1,...,1}_{|U_5|}, \underbrace{2,2,...,2}_{|U_6|}, \underbrace{2,2,...,2}_{|U_7|}, 3,3,...,3)$,
$j^{th}$ coordinate is 0,
$\zeta(a_{[3][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_4|}, \underbrace{2,2,...,2}_{|U_5|}, \underbrace{2,2,...,2}_{|U_6|}, \underbrace{2,2,...,2}_{|U_7|}, 3,3,...,3)$,
$j^{th}$ coordinate is 0,
$\zeta(a_{[4][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_3|}, \underbrace{2,2,...,2}_{|U_4|-|U_3|}, \underbrace{1,1,...,1}_{|U_5|}, \underbrace{1,1,...,1}_{|U_6|}, \underbrace{1,1,...,1}_{|U_7|}, 2,2,...,2)$,
$j^{th}$ coordinate is 0,
$\zeta(a_{[5][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_2|}, \underbrace{2,2,...,2}_{|U_4|+|U_5|-|U_2|}, \underbrace{1,1,...,1}_{|U_6|}, \underbrace{1,1,...,1}_{|U_7|}, 2,2,...,2)$,
$(j+|U_4|)^{th}$ coordinate is 0,
$\zeta(a_{[6][j]}|\Pi) = (1,1,...,1)$, $(j+|U_4|+|U_5|)^{th}$ coordinate is 0,
$\zeta(a_{[7][j]}|\Pi) = (1,1,...,1)$, $(j+|U_4|+|U_5|+|U_6|)^{th}$ coordinate is 0.

By the above representation, $\zeta(a_{[i][j]}|\Pi)$ has a unique partition representation. Hence, $\Pi$ is a resolving partition and $pd(\Lambda(\mathbb{Z}_m)) \leq l_1^{s-1}(l_2-1)(l_3-1)$.

Since $|U_1| = l_1^{s-1}(l_2-1)(l_3-1)$ and by Observation 2.1, $pd(\Lambda(\mathbb{Z}_m)) \geq l_1^{s-1}(l_2-1)(l_3-1)$. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1}(l_2-1)(l_3-1)$ and this proves the result.

**Proposition 4.2.** *Let $m = l_1^s l_2 l_3$ where $l_1, l_2$ and $l_3$ are prime numbers with $l_1 > l_2 > l_3, l_1, l_2, l_3 \geq 2, s \geq 2$, then $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1}(l_1-1)(l_2-1)$.*

**Proof.** If $m = l_1^s l_2 l_3, l_1 > l_2 > l_3$, then the vertices can be partitioned into 7 distance similar equivalence classes as

$U_1 = \{tl_3 < m \mid t \in \mathbb{N}\} \setminus \bigcup_{i=2}^{7} U_i$,

$U_2 = \{tl_2 < m \mid t \in \mathbb{N}\} \backslash \bigcup_{i=1, i\neq 2}^{7} U_i,$

$U_3 = \{tl_1 < m \mid t \in \mathbb{N}\} \backslash \bigcup_{i=1, i\neq 3}^{7} U_i,$

$U_4 = \{tl_1 l_2 < m \mid t \in \mathbb{N}\} \backslash U_7,$

$U_5 = \{tl_1 l_3 < m \mid t \in \mathbb{N}\} \backslash U_7,$

$U_6 = \{tl_2 l_3 < m \mid t \in \mathbb{N}\} \backslash U_7,$

$U_7 = \{tl_1 l_2 l_3 < m \mid t \in \mathbb{N}\}.$

Let us denote $U_i = \{a_{[i][j]}\}, 1 \leq i \leq 7, 1 \leq j \leq |U_i|$. Consider the set of partition $\Pi = \{S_1, S_2, ..., S_{l_1^{s-1}(l_1-1)(l_2-1)}\}$.

$S_1 = \{a_{[1][1]}, a_{[2][1]}, a_{[3][1]}, a_{[4][1]}\},$

$S_2 = \{a_{[1][2]}, a_{[2][2]}, a_{[3][2]}, a_{[4][2]}\},$

...

$S_{|U_4|} = \{a_{[1][|U_4|]}, a_{[2][|U_4|]}, a_{[3][|U_4|]}, a_{[4][|U_4|]}\},$

$S_{|U_4|+1} = \{a_{[1][|U_4|+1]}, a_{[2][|U_4|+1]}, a_{[3][|U_4|+1]}, a_{[5][1]}\},$

...

$S_{|U_4|+|U_5|} = \{a_{[1][|U_4|+|U_5|]}, a_{[2][|U_4|+|U_5|]}, a_{[3][|U_4|+|U_5|]}, a_{[5][|U_5|]}\},$

...

$S_{l_1^{s-1}(l_1-1)(l_2-1)} = \{a_{[1][l_1^{s-1}(l_1-1)(l_2-1)]}\}.$

The partition representation of $a_{[i][j]}$ with respect to the set $\Pi$ is given by

$\zeta(a_{[1][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_4|}, \underbrace{2, 2, ..., 2}_{|U_5|}, \underbrace{2, 2, ..., 2}_{|U_6|}, \underbrace{1, 1, ..., 1}_{|U_7|}, 2, 2, ..., 2),$

$j^{th}$ coordinate is 0,

$\zeta(a_{[2][j]}|\Pi) = (\underbrace{2, 2, ..., 2}_{|U_4|}, \underbrace{1, 1, ..., 1}_{|U_5|}, \underbrace{2, 2, ..., 2}_{|U_6|}, \underbrace{2, 2, ..., 2}_{|U_7|}, 3, 3, ..., 3),$

$j^{th}$ coordinate is 0,

$\zeta(a_{[3][j]}|\Pi) = (\underbrace{2, 2, ..., 2}_{|U_4|}, \underbrace{2, 2, ..., 2}_{|U_5|}, \underbrace{1, 1, ..., 1}_{|U_6|}, \underbrace{2, 2, ..., 2}_{|U_7|}, 3, 3, ..., 3),$

$j^{th}$ coordinate is 0,

$\zeta(a_{[4][j]}|\Pi) = (1, 1, ..., 1),$ $j^{th}$ coordinate is 0,

$\zeta(a_{[5][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_2|}, \underbrace{2, 2, ..., 2}_{|U_4|+|U_5|-|U_2|}, \underbrace{1, 1, ..., 1}_{|U_6|}, \underbrace{1, 1, ..., 1}_{|U_7|}, 2, 2, ..., 2),$

$(j + |U_4|)^{th}$ coordinate is 0,

$\zeta(a_{[6][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_4|}, \underbrace{1, 1, ..., 1}_{|U_5|}, \underbrace{2, 2, ..., 2}_{|U_4|+|U_5|+|U_6|-|U_3|}, \underbrace{1, 1, ..., 1}_{|U_7|}, 2, 2, ..., 2),$

$(j + |U_4| + |U_5|)^{th}$ coordinate is 0,

$\zeta(a_{[7][j]}|\Pi) = (1, 1, ..., 1),$ $(j + |U_4| + |U_5| + |U_6|)^{th}$ coordinate is 0.

By the above representation, $\zeta(a_{[i][j]}|\Pi)$ has a unique partition representation. Hence, $\Pi$ is a resolving partition and $pd(\Lambda(\mathbb{Z}_m)) \leq l_1^{s-1}(l_1-1)(l_2-1)$.

Since $|U_1| = l_1^{s-1}(l_1-1)(l_2-1)$ and by Observation 2.1, $pd(\Lambda(\mathbb{Z}_m)) \geq l_1^{s-1}(l_1-1)(l_2-1)$. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1^{s-1}(l_1-1)(l_2-1)$ and this proves the result.

In the following results, we calculate the partition dimension of $\mathbb{Z}_m$ for $m = l_1^2 l_2^2 l_3$ using three conditions when $l_1 < l_2 < l_3, l_1 < l_3 < l_2$ and $l_1 > l_2 > l_3$.

**Proposition 4.3.** *Let $m = l_1^2 l_2^2 l_3$, where $l_1, l_2$ and $l_3$ are primes with $l_1 < l_2 < l_3, l_1, l_2, l_3 \geq 2$, then $pd(\Lambda(\mathbb{Z}_m)) = l_1 l_2 (l_2-1)(l_3-1)$.*

**Proof.** If $m = l_1^2 l_2^2 l_3, l_1 < l_2 < l_3$, then the vertices can be partitioned into 9 distance similar equivalence classes as

$U_1 = \{tl_1 < m \mid t \in \mathbb{N}\} \backslash \bigcup_{i=2}^{9} U_i$,

$U_2 = \{tl_2 < m \mid t \in \mathbb{N}\} \backslash \bigcup_{i=1, i\neq 2}^{9} U_i$,

$U_3 = \{tl_3 < m \mid t \in \mathbb{N}\} \backslash \bigcup_{i=1, i\neq 3}^{9} U_i$,

$U_4 = \{tl_1 l_2 < m \mid t \in \mathbb{N}\} \backslash U_7 \cup U_8 \cup U_9$,

$U_5 = \{tl_1 l_3 < m \mid t \in \mathbb{N}\} \backslash U_7 \cup U_8 \cup U_9$,

$U_6 = \{tl_2 l_3 < m \mid t \in \mathbb{N}\} \backslash U_7 \cup U_8 \cup U_9$,

$U_7 = \{tl_1 l_2 l_3 < m \mid t \in \mathbb{N}\} \backslash U_8 \cup U_9$,

$U_8 = \{tl_1^2 l_2 l_3 < m \mid t \in \mathbb{N}\}$,

$U_9 = \{tl_1 l_2^2 l_3 < m \mid t \in \mathbb{N}\}$.

Let us denote $U_i = \{a_{[i][j]}\}, 1 \leq i \leq 9, 1 \leq j \leq |U_i|$. Consider the set of partition $\Pi = \{S_1, S_2, ..., S_{l_1^{s-1} l_2^{s-1}(l_2-1)(l_3-1)}\}$.

$S_1 = \{a_{[1][1]}, a_{[2][1]}, a_{[3][1]}, a_{[4][1]}\}$,

$S_2 = \{a_{[1][2]}, a_{[2][2]}, a_{[3][2]}, a_{[4][2]}\}$,

...

$S_{|U_4|} = \{a_{[1][|U_4|]}, a_{[2][|U_4|]}, a_{[3][|U_4|]}, a_{[4][|U_4|]}\}$,

$S_{|U_4|+1} = \{a_{[1][|U_4|+1]}, a_{[2][|U_4|+1]}, a_{[3][|U_4|+1]}, a_{[5][1]}\}$,

...

$S_{|U_4|+|U_5|} = \{a_{[1][|U_4|+|U_5|]}, a_{[2][|U_4|+|U_5|]}, a_{[3][|U_4|+|U_5|]}, a_{[5][|U_5|]}\}$,

...

$S_{l_1 l_2(l_2-1)(l_3-1)} = \{a_{[1][l_1 l_2(l_2-1)(l_3-1)]}\}$.

The partition representation of $a_{[i][j]}$ with respect to the set $\Pi$ is given by

$\zeta(a_{[1][j]}|\Pi) = (\underbrace{2, 2, ..., 2}_{|U_4|+|U_5|}, \underbrace{1, 1, ..., 1}_{|U_6|}, \underbrace{2, 2, ..., 2}_{|U_7|+|U_8|}, \underbrace{1, 1, ..., 1}_{|U_9|}, 2, 2, ..., 2)$,

$j^{th}$ coordinate is 0,

$\zeta(a_{[2][j]}|\Pi) = (\underbrace{2,2,...,2}_{|U_4|}, \underbrace{1,1,...,1}_{|U_5|}, \underbrace{2,2,...,2}_{|U_6|+|U_7|}, \underbrace{1,1,...,1}_{|U_8|}, \underbrace{2,2,...,2}_{|U_9|}, 3,3,...,3)$,
$j^{th}$ coordinate is 0,
$\zeta(a_{[3][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_4|}, \underbrace{2,2,...,2}_{|U_5|+|U_6|+...+|U_9|}, 3,3,...,3)$, $j^{th}$ coordinate is 0,
$\zeta(a_{[4][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_3|}, \underbrace{2,2,...,2}_{|U_4|-|U_3|}, \underbrace{1,1,...,1}_{|U_5|+|U_6|+...+|U_9|}, 2,2,...,2)$, $j^{th}$ coordinate is 0,
$\zeta(a_{[5][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_2|}, \underbrace{2,2,...,2}_{|U_4|+|U_5|-|U_2|}, \underbrace{1,1,...,1}_{|U_6|}, \underbrace{2,2,...,2}_{|U_7|+|U_8|}, \underbrace{1,1,...,1}_{|U_9|}, 2,2,...,2)$,
$(j+|U_4|)^{th}$ coordinate is 0,
$\zeta(a_{[6][j]}|\Pi) = (1,1,...,1)$, $(j+|U_4|+|U_5|)^{th}$ coordinate is 0,
$\zeta(a_{[7][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_4|}, \underbrace{2,2,...,2}_{|U_5|+|U_6|}, \underbrace{1,1,...,1}_{|U_7|+|U_8|+|U_9|}, 2,2,...,2)$,
$(j+\sum_{i=4}^{6}|U_i|)^{th}$ coordinate is 0,
$\zeta(a_{[8][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_2|}, \underbrace{2,2,...,2}_{|U_4|+|U_5|-|U_2|}, \underbrace{1,1,...,1}_{|U_6|+|U_7|+|U_8|+|U_9|}, 2,2,...,2)$,
$(j+\sum_{i=4}^{7}|U_i|)^{th}$ coordinate is 0,
$\zeta(a_{[9][j]}|\Pi) = (1,1,...,1)$, $(j+\sum_{i=4}^{8}|U_i|)^{th}$ coordinate is 0.

By the above representation, $\zeta(a_{[i][j]}|\Pi)$ has a unique partition representation. Hence, $\Pi$ is a resolving partition and $pd(\Lambda(\mathbb{Z}_m)) \le l_1l_2(l_2-1)(l_3-1)$.

Since $|U_1| = l_1l_2(l_2-1)(l_3-1)$ and by Observation 2.1, $pd(\Lambda(\mathbb{Z}_m)) \ge l_1l_2(l_2-1)(l_3-1)$. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1l_2(l_2-1)(l_3-1)$ and this proves the result.

**Proposition 4.4.** *Let $m = l_1^2l_2^2l_3$, where $l_1, l_2$ and $l_3$ are primes with $l_1 < l_3 < l_2, l_1, l_2, l_3 \ge 2$, then $pd(\Lambda(\mathbb{Z}_m)) = l_1l_2(l_1-1)(l_3-1)$.*
**Proof.** If $m = l_1^2l_2^2l_3, l_1 < l_3 < l_2$, then the vertices can be partitioned into 9 distance similar equivalence classes as in Preposition 4.3.
Let us denote $U_i = \{a_{[i][j]}\}, 1 \le i \le 9, 1 \le j \le |U_i|$. Consider the set of partition $\Pi = \{S_1, S_2, ..., S_{l_1^{s-1}l_2^{s-1}(l_2-1)(l_3-1)}\}$.
$S_1 = \{a_{[1][1]}, a_{[2][1]}, a_{[3][1]}, a_{[4][1]}\}$,
$S_2 = \{a_{[1][2]}, a_{[2][2]}, a_{[3][2]}, a_{[4][2]}\}$,
...
$S_{|U_4|} = \{a_{[1][|U_4|]}, a_{[2][|U_4|]}, a_{[3][|U_4|]}, a_{[4][|U_4|]}\}$,
$S_{|U_4|+1} = \{a_{[1][|U_4|+1]}, a_{[2][|U_4|+1]}, a_{[3][|U_4|+1]}, a_{[5][1]}\}$,

...
$S_{|U_4|+|U_5|} = \{a_{[1][|U_4|+|U_5|]}, a_{[2][|U_4|+|U_5|]}, a_{[3][|U_4|+|U_5|]}, a_{[5][|U_5|]}\}$,
...
$S_{l_1 l_2(l_1-1)(l_3-1)} = \{a_{[1][l_1 l_2(l_1-1)(l_3-1)]}\}$. The partition representation of $a_{[i][j]}$ with respect to the set $\Pi$ is same as in the Preposition 4.3 except for $\zeta(a_{[4][j]}|\Pi)$.
$\zeta(a_{[4][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_3|}, \underbrace{1,1,...,1}_{|U_4|+|U_5|-|U_3|}, \underbrace{1,1,...,1}_{|U_6|+...+|U_9|}, 2,2,...,2)$, $j^{th}$ coordinate is 0.

By the above representation, $\zeta(a_{[i][j]}|\Pi)$ has a unique partition representation. Hence, $\Pi$ is a resolving partition and $pd(\Lambda(\mathbb{Z}_m)) \leq l_1 l_2(l_1-1)(l_3-1)$.

Since $|U_1| = l_1 l_2(l_1-1)(l_3-1)$ and by Observation 2.1, $pd(\Lambda(\mathbb{Z}_m)) \geq l_1 l_2(l_1-1)(l_3-1)$. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1 l_2(l_1-1)(l_3-1)$ and this proves the result.

**Proposition 4.5.** *Let $m = l_1^2 l_2^2 l_3$, where $l_1, l_2$ and $l_3$ are primes with $l_1 > l_2 > l_3, l_1, l_2, l_3 \geq 2$, then $pd(\Lambda(\mathbb{Z}_m)) = l_1 l_2(l_1-1)(l_2-1)$.*
**Proof.** If $m = l_1^2 l_2^2 l_3, l_1 > l_2 > l_3$, then the vertices can be partitioned into 9 distance similar equivalence classes as in Preposition 4.3.
Let us denote $U_i = \{a_{[i][j]}\}, 1 \leq i \leq 9, 1 \leq j \leq |U_i|$. Consider the set of partition $\Pi = \{S_1, S_2, ..., S_{l_1^{s-1} l_2^{s-1}(l_2-1)(l_3-1)}\}$.
$S_1 = \{a_{[3][1]}, a_{[2][1]}, a_{[1][1]}, a_{[4][1]}\}$,
$S_2 = \{a_{[3][2]}, a_{[2][2]}, a_{[1][2]}, a_{[4][2]}\}$,
...
$S_{|U_4|} = \{a_{[3][|U_4|]}, a_{[2][|U_4|]}, a_{[1][|U_4|]}, a_{[4][|U_4|]}\}$,
$S_{|U_4|+1} = \{a_{[3][|U_4|+1]}, a_{[2][|U_4|+1]}, a_{[1][|U_4|+1]}, a_{[5][1]}\}$,
...
$S_{|U_4|+|U_5|} = \{a_{[3][|U_4|+|U_5|]}, a_{[2][|U_4|+|U_5|]}, a_{[1][|U_4|+|U_5|]}, a_{[5][|U_5|]}\}$,
...
$S_{l_1 l_2(l_1-1)(l_2-1)} = \{a_{[3][l_1 l_2(l_1-1)(l_2-1)]}\}$.

The partition representation of $a_{[i][j]}$ with respect to the set $\Pi$ is given by
$\zeta(a_{[1][j]}|\Pi) = (\underbrace{2,2,...,2}_{|U_4|+|U_5|}, \underbrace{1,1,...,1}_{|U_6|}, \underbrace{2,2,...,2}_{|U_7|+|U_8|}, \underbrace{1,1,...,1}_{|U_9|}, 3,3,...,3)$,
$j^{th}$ coordinate is 0,
$\zeta(a_{[2][j]}|\Pi) = (\underbrace{2,2,...,2}_{|U_4|}, \underbrace{1,1,...,1}_{|U_5|}, \underbrace{2,2,...,2}_{|U_6|+|U_7|}, \underbrace{1,1,...,1}_{|U_8|}, \underbrace{2,2,...,2}_{|U_9|}, 3,3,...,3)$,
$j^{th}$ coordinate is 0,
$\zeta(a_{[3][j]}|\Pi) = (\underbrace{1,1,...,1}_{|U_4|}, \underbrace{2,2,...,2}_{|U_5|+|U_6|+...+|U_9|}, 2,2,...,2)$, $j^{th}$ coordinate is 0,
$\zeta(a_{[4][j]}|\Pi) = (1,1,...,1)$, $j^{th}$ coordinate is 0,

$\zeta(a_{[5][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_2|}, \underbrace{1, 1, ..., 1}_{|U_4|+|U_5|+|U_6|-|U_2|}, \underbrace{2, 2, ..., 2}_{|U_7|+|U_8|}, \underbrace{1, 1, ..., 1}_{|U_9|}, 2, 2, ..., 2)$,
$(j + |U_4|)^{th}$ coordinate is 0,
$\zeta(a_{[6][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_4|+|U_5|}, \underbrace{2, 2, ..., 2}_{|U_6|+|U_7|}, \underbrace{1, 1, ..., 1}_{|U_8|}, \underbrace{2, 2, ..., 2}_{|U_9|}, 2, 2, ..., 2)$, $(j + |U_4| + |U_5|)^{th}$ coordinate is 0,
$\zeta(a_{[7][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_4|}, \underbrace{2, 2, ..., 2}_{|U_5|+|U_6|}, \underbrace{1, 1, ..., 1}_{|U_7|+|U_8|+|U_9|}, 2, 2, ..., 2)$,
$(j + \sum_{i=4}^{6} |U_i|)^{th}$ coordinate is 0,
$\zeta(a_{[8][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_2|}, \underbrace{1, 1, ..., 1}_{|U_4|+|U_5|+|U_6|-|U_2|}, \underbrace{1, 1, ..., 1}_{|U_7|+|U_8|+|U_9|}, 2, 2, ..., 2)$,
$(j + \sum_{i=4}^{7} |U_i|)^{th}$ coordinate is 0,
$\zeta(a_{[9][j]}|\Pi) = (\underbrace{1, 1, ..., 1}_{|U_4|+|U_5|+...+|U_9|}, 2, 2, ..., 2)$, $(j + \sum_{i=4}^{8} |U_i|)^{th}$ coordinate is 0.

By the above representation, $\zeta(a_{[i][j]}|\Pi)$ has a unique partition representation. Hence, $\Pi$ is a resolving partition and $pd(\Lambda(\mathbb{Z}_m)) \leq l_1 l_2(l_1 - 1)(l_2 - 1)$.

Since $|U_3| = l_1 l_2(l_1 - 1)(l_2 - 1)$ and by Observation 2.1, $pd(\Lambda(\mathbb{Z}_m)) \geq l_1 l_2(l_1 - 1)(l_2 - 1)$. Therefore, $pd(\Lambda(\mathbb{Z}_m)) = l_1 l_2(l_1 - 1)(l_2 - 1)$ and this proves the result.

## 5. Conclusion

In this paper, we have investigated the partition dimension of the extended zero divisor graphs of some finite commutative rings particularly some cases of ring of integers modulo $m$. To investigate partition dimension in other cases like multiple product of prime powers in the ring of integers modulo $m$ and establish similar results for other graphs like total graph, unit graph, line graphs etc,. is an open area of research.

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## References

[1] Atiyah, M. F., & Macdonald, I. G., Introduction to commutative algebra, Addison, 1969.

[2] Bennis, D., Mikram, J., Taraza, F., On the extended zero divisor graph of commutative rings, Turkish Journal of Mathematics, 40(2) (2016), 376-388.

[3] Chartrand, G., Salehi, E., Zhang, P., On the partition dimension of a graph, Congr. Numer., 131 (2000), 55–66.

[4] Chartrand, G., Eroh, L., Johnson, M. A., Oellermann, O. R., Resolvability in graphs and the metric dimension of a graph. Discrete Applied Mathematics, 105(1-3) (2000), 99-113.

[5] Chartrand, G., Salehi, E., Zhang, P., The partition dimension of a graph, Aequationes mathematicae, 59(1) (2000), 45-54.

[6] Diestel, R., Graph theory, Springer-Verlag, New York, 1997.

[7] Fernau, H., Rodríguez-Velázquez, J. A. & Yero, I. G., On the partition dimension of unicyclic graphs. Bulletin mathématique de la Société des Sciences Mathématiques de Roumanie, (2014), 381-391.

[8] Grigorious, C., Stephen, S., Rajan, B. & Miller, M., On the partition dimension of circulant graphs, The Computer Journal, 60(2) (2017), 180-184.

[9] Javaid, I., Raja, N. K., Salman, M. & Azhar, M. N., The partition dimension of circulant graphs, World Applied Sciences Journal, 18(12) (2012), 1705-1717.

[10] Kaplansky, I., Commutative Rings, University of Chicago Press, Chicago, 1974.

[11] Melter, R. A., Tomescu, I., Metric bases in digital geometry, Computer vision, graphics, and image Processing, 25(1) (1984), 113-121.

[12] Nithya, S., & Elavarasi, G. Extended annihilating-ideal graph of a commutative ring, Discussiones Mathematicae-General Algebra and Applications, 42(2), 279-291.

[13] Nithya, S., & Elavarasi, G., Partition and local metric dimension of an extended annihilating-ideal graph, Palestine Journal of Mathematics, 11 (2022).

[14] Nithya, S., & Prisci, V., On the metric dimension of extended zero divisor graphs. Gulf Journal of Mathematics, 16(1) (2024), 205-214.

[15] Rodríguez-Velázquez, J. A., Yero, I. G. & Lemańska, M., On the partition dimension of trees, Discrete Applied Mathematics, 166 (2014), 204-209.

[16] Slater, P. J., Leaves of trees. Proceeding of the 6th Southeastern Conference on Combinatorics, Graph Theory, and Computing, Congressus Numerantium, 14 (1975), 549-559.

[17] Tomescu, I., Discrepancies between metric dimension and partition dimension of a connected graph, Discrete Mathematics, 308(22) (2008), 5026-5031.

[18] Tomescu, I., Javaid, I. & Slamin, I., On the partition dimension and connected partition dimension of wheels, Ars combinatoria, 84 (2007), 311-318.

[19] West, D. B., Introduction to graph theory (Vol. 2), Upper Saddle River: Prentice hall, 2001.